APOS Theory towards Algebraic Thinking Skill

Fevi Rahmawati Suwanto[1], Aprisal[2], Misnasanti[3], Wan Denny Pramana Putra[4], Rospala Hanisah Yukti Sari[5]
Yogyakarta State University

E-mail: fevirahmawatisuwanto@gmail.com

Abstract. Future education requires knowledge and technology for students to face life's challenges. One of the basic science that supports the purpose of education is mathematics. Mathematics is the basic science that supports the development of other sciences. In preparing students to be success in mathematics, it is required a grip that becomes the basis for developing students' skills in math. Algebraic thinking skill is one of the basic skills that support students to develop their math skills. Algebraic thinking skill is the ability to think using algebraic symbols. Algebraic thinking skill involves activities that consist of generational, transformational, and global meta-levels. Algebraic thinking skill appears as a representation of the students' ability to learn and understand the material of school algebra. Algebra is a branch of mathematics as a gateway to the future of technology. Algebra is very important for students because it is used in everyday life either implicitly or explicitly. In algebra thinking, it is needed a high cognitive ability so that the cognitive development of students is an aspect that needs to be considered in the process of learning achievement of mathematics. A theory which supports students' cognitive development is APOS Theory. APOS theory is oriented towards social constructivism that adopted by Vygotsky. This theory that developed by Dubinsky can build knowledge and understanding of mathematics through mental construction in the form of action, process, object, and scheme. Therefore, the purpose of the literature review in this article is to examine, analyze, and describe the results of studies related to APOS Theory on students' algebraic thinking skills.

1. Introduction

In general, algebra is a branch of mathematics that studies abstract structures and manipulations in symbolic form to solve life problems. The problem is not only derived from in mathematics, but also from other areas of knowledge that can be solved algebraically. This is in line with [1] which states that "algebraic competence is important in adult life, both on the job and as preparation for postsecondary education. All students should learn algebra". Therefore, students must master the competence of algebra as a provision in the face of life.

The importance of students to mastering algebra makes it a goal of the curriculum since prekindergarten. Teaching that seeks to train students to think algebra from an early age provides an opportunity for teachers to help students build a strong foundation of understanding and experience in algebra at the secondary and high school levels [1]. Although it has been done since the early age, in reality, there are still many students who have difficulty in the operation of algebra or solve problems associated with algebra.

According to [2] conducted research on students' difficulties in studying algebraic forms related to concepts and principles. From the research result, it is found that students have difficulty using concept and principle when studying algebraic form. The difficulty of using concepts can be difficult to identify: (1) coefficients, (2) variables, (3) similar tribes. While the difficulty of using the principle may be: (1) performing the addition and subtraction operation of the negative value tribe, (2) determining the result of the addition operation and the reduction of the unbalanced tribe, (3) determining the result of the reduction of the first term with the two, and (4) do multiplicative distributive properties in addition.

Then [3] conducted research on student difficulties in solving algebraic problems. The results of this study indicate the types of difficulties experienced by students in solving algebra arithmetical problems. This type of difficulty can be classified into three types: difficulties in transferring knowledge, difficulty in understanding the mathematical language, and difficulty in calculating.

From some of these studies, many students have difficulty using algebra in studying and solving problems. In other words, students still have difficulty in learning mathematical concepts in algebraic form. This indicates that there are still many students who do not have algebraic thinking skills.

Given the importance of algebra and algebraic thinking as a provision in dealing with various problems of life, it is necessary to have an alternative or theory that can describe and even increase students' algebra thinking, one of them is APOS. The APOS theory can be used even successfully in many studies as a developmental perspective, as an evaluation tool, or both. For example, based on research [4] obtained the result that by using APOS, students did not appear to suffer from the misconception that a function has to be defined by an algebraic formula. Then in the research of [5], it was found that APOS theory as a tool for understanding students' difficulties with infinite processes in mathematics and for suggesting effective pedagogical strategies to help students overcome these difficulties.

2. Discussion

2.1. APOS Theory

APOS (Action, Process, Object, and Schema) is a theory that is closely related to the way to learn a mathematical concept. Dubinsky's developed theory is a form of Piaget's reflection abstraction in the form of a schematic on postsecondary math conducted since about 1983 [6]. The development of this theory is based on cognitive development aimed at helping students learn and how teachers guide students to learn.

In essence, this theory is epistemological and psychological because it is closely related to the nature of mathematical concepts and its development in the minds of students [7]. This leads to the implementation of APOS theory in mathematics learning into a developmental scheme theory that requires students to think logically and mathematically. In line with that [8] reveals that the theory of APOS present as one means to illustrate the logical thinking ability of children and develop it to a more complex idea of mathematical concepts.

APOS theory is a theory with a constructivist approach on how to learn mathematical concepts [9]. The purpose of mathematical concepts studied using this theory is to become more meaningful [6]. Thus, the implementation of APOS theory in the learning process is encouraging students to construct their own knowledge of mathematical concepts through a series of activities.

A person's mathematical concept begins to form when applying a transformation to an object to get another object. The first transformation is understood as an action, it because requires specific instruction and the ability to perform each step of the transformation explicitly. Students repeat and reflect an inter-institutionalized action into a mental process. A process is a mental structure that forms the same operation with inspired action but is entirely within the individual's mind. This allows students to imagine transforming without having to do each step explicitly. If the student is aware of the process as a totality, in which the transformation can act and built either explicitly or imaginatively, then it can be said the student has put this process into the cognitive object. Activity proposes the process through action to be cognitive objective called encapsulation. Further encapsulation allows one to apply a transformation. Mathematical concepts often involve many actions, processes, and objects that need to be organized and connected into a coherent framework called a scheme. The coherent framework here means the framework that provides the student to determine the mental structure to be used in dealing with mathematical situations. Thus, in this theory,

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students will use certain mental mechanisms in building cognitive structures when faced with mathematical situations. The main mental mechanism is called interiorization and encapsulation and its associated structures are actions, processes, objects, and schemes [10].

The framework of APOS theory in mathematics learning refers to physical and mental activity. In more detail, [11] describe the process flow of mental characteristics of an APOS builder structure based on actions, processes, objects, and schemes and mechanisms such as interiozation, encapsulation, coordination, reversal, de-encapsulation, schematization, and generalization of the activity through the following diagram.

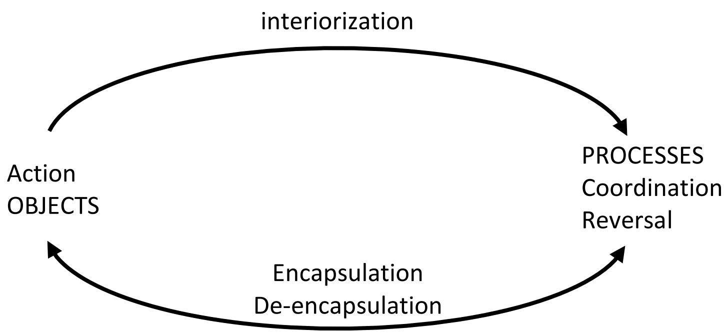

Figure 1. Construction on Mathematical Knowledge

A concept is first conceived as an Action, that is, as an externally directed transformation of a previously conceived Object, or Objects. An Action is external in the sense that each step of the transformation needs to be performed explicitly and guided by external instructions; additionally, each step prompts the next, that is, the steps of the Action cannot yet be imagined and none can be skipped**.** Processes are constructed using one of two mental mechanisms: interiorization or coordination. Each of these mechanisms gives rise to new Processes. Encapsulation occurs when an individual applies an Action to a Process, that is, sees a dynamic structure (Process) as a static structure to which Actions can be applied. Once a Process has been encapsulated into a mental Object, it can be de-encapsulated, when the need arises, back to its underlying Process. In other words, by applying the mechanism of de-encapsulation, an individual can go back to the Process that gave rise to the Object. The mechanism of coordination is indispensable in the construction of some Objects. Two Objects can be de-encapsulated, their Processes coordinated, and the coordinated Process encapsulated to form a new Object. The coherence of a Schema is determined by the individual's ability to ascertain whether it can be used to deal with a particular mathematical situation. Once a Schema is constructed as a coherent collection of structures (Actions, Processes, Objects, and other Schemas) and connections established among those structures, it can be transformed into a static structure (Object) and/or used as a dynamic structure that assimilates other related Objects or Schemas [6]. Meanwhile, [8] outlines the characteristics of the APOS framework in mathematics learning, as follows.

Table 1. Characteristics of APOS

Framework of APOS	Characteristics
Action	a) Activities involving procedural matters b) Focus on the algorithm in solving the problem c) Tend to solve the problem according to the given example d) Only apply the mathematical concepts of formal formulas according to the example e) Require detailed guidance to solve the problem (transformation)

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Process	a) It does not require a guide to transform b) Be able to explain structured steps of transformation without actually doing it c) This activity is a procedural understanding
Object	a) Be able to perform actions and processes against mathematical objects encountered b) This process is a conceptual understanding c) Be able to explain logically and structurally the transformation is done d) Be able to explain the properties of mathematical concepts
Schema	a) Be able to connect various concepts of mathematics or transformation of concepts on the activities of action, process, or object b) Can connect actions, processes, objects with other previously known properties c) Understand the formulas needed to solve the problem

2.2. Algebraic Thinking Skill

Algebra is one of the scopes of mathematical material that has an important role. Almost all material in mathematics requires an algebraic process to solve existing problems, such as on statistical content and opportunities, geometry and measurements, trigonometry, and calculus. [12] explains that on of the key to successful students in learning algebra is to develop their algebraic thinking skills. Algebraic thinking according to Mason, et al [13] is actually owned by every child who starts school by demonstrating the ability to generalize and abstract on certain cases. At the elementary level, students have the motivation and curiosity to learn math by describing and extending patterns of shapes, colors, sounds, and finally on letters and numbers. At a later stage, students have been able to make generalizations about patterns that appear to be the same or different. This kind of activity becomes an important step to lead students to algebraic thinking skills.

Algebraic thinking is a representation of the students' ability to learn algebra and use algebraic symbols such as equations, inequalities, and so on. [14] explains that algebraic thinking is a form of mathematical reasoning that consists of generalizations, abstractions of calculations, and relationships between variables or mathematical concepts. Meanwhile, Mc Clure [15] defines algebraic thinking as a way of thinking of students in analyzing the given problem so that it can be generalized, modeled, justified, proved, and problem-solving. From these various opinions, a student can be said to be thinking algebra when he is able to do mathematical reasoning consisting of generalization, modeling, justification, verification, and solving problems given in the form of representation using algebraic symbols.

An important aspect of algebraic thinking is the ability to generalize from the given cases [13]. Similarly, [16] argues that algebraic thinking or algebraic reasoning involves the formation of generalizations based on the computational results of numbers that have been done by students, using algebraic notation or symbols, and identifying concepts that match the observed patterns. Thus, algebraic thinking skill is the ability to generalize mathematical patterns, look at relationships between mathematical variables, and form mathematical models to solve problems.

According to [17], there are three indicators that show students' algebraic thinking skills, namely: generational activities, transformational activities, and global meta-level activities. Generational activities are activities that include the formation of algebraic object expressions and equation problems. The expression of algebraic objects can be numerical sequence patterns, geometric patterns, and the use of formulas that are related to numerical solution, whereas the equation problem can be the use and meaning of the sign equal to, the solution of the equation. Transformational activities are algebraic expression changes that can be factoring, extending, substituting, performing addition operations, subtraction, multiplication, and division of two polynomials, simplifying expressions, and changing expressions to equivalent expressions. Finally, meta-global level activities are activities to solve algebraic problems and problems beyond algebra using algebra itself. These activities include analyzing relationships and change, math modeling, problem-solving, problem-solving and science-problem solving using algebra. The global meta-level activity becomes an important aspect for students, especially the middle level because one of the algebraic concept functions that students have learned can be applied in problem-solving [18].

Meanwhile, [19] states that there are five aspects of algebraic thinking: generalizations of arithmetic and patterns, meaningful use of symbols, making structures in number systems explicitly, examining patterns and functions, and modeling mathematics. Kaput [19] also gives five indicators in algebraic thinking, namely generalizing from arithmetic and patterns, using meaningful symbols, studying structures in number systems, studying patterns and functions, and doing mathematical modeling.

2.3. APOS Theory towards Algebraic Thinking

Based on the results of [20] study that learning by using the theory of APOS able to develop students' ability invalidating evidence compared with students who obtained learning by conventional methods. The ability to validate evidence according to [21] includes the ability to use definitions, entries, and theorems to solve problems related to proof. The statement is in line with the wrong meta-global level indicator on algebraic thinking ability that students are able to solve the problem of proof [17].

Meanwhile, the study by [22] showed significant results that the learning based on APOS theory with ACE syntax approach increased the students' understanding of the concept of 8th grade SMP N 3 Polokarto Sukoharjo on SPLDV material. Indicators of understanding of concepts used in the study: (1) ability in understanding the problem, (2) ability in solving problems, and (3) ability to re-state an SPLDV concept. If examined in depth, then the indicator is an intersection with algebra thinking indicator according to [17] at the level of transformational and meta-global level. Both studies show that learning based on APOS theory is able to develop and improve students' algebraic thinking ability.

Based on the results of research Zahid, [23], APOS theory can describe the construction process of students' knowledge in doing factorization on algebra. At the stage of action, students are able to name what is known from the problem in different ways in the form of images, or words. At the stage of the process, the student is able to reflect, explain, or even reverse the step of the transformation on the previously studied object without actually performing the step by multiplication of the algebraic appropriately. At the object stage, students are able to associate, reverse, and decompose the process in terms of algebraic multiplication. Finally, at the stage of the scheme, students are able to explain the factors obtained through factorization if multiplied will result in the algebraic form searched for factors or in other words the students are able to connect the process and the object through which to form a complete understanding of the concept of algebraic factors.

Another study conducted by [24] which aims to explore students' algebraic thinking through a semiotic approach based on APOS theory categorizes students' algebra thinking process into four groups: (1) factual, (2) contextual, (3) symbolic short jumping, and (4) symbolic high jumping. Factual algebraic thinking includes activities to generalize patterns in which the process is more dominant in this category. Generalization results of the pattern expressed in terms of words or sentences based on the basic rules. Contextual algebraic thinking when in generalization activity more dominant pattern is action activity, while at process stage not so complete that at stage object also become incomplete. Algebraic thinking symbolic short jumping is the generalization of the pattern when the stage of the process and the stage of the object is complete and perfect. Thinking symbolic high jumping algebra occurs when the activity of generating where from action stage, process, and object to organizing it into the scheme is done perfectly.

3. Conclusion

From the description above, it can be concluded that algebraic thinking is the ability to solve various mathematical problems or other related fields of knowledge by representing them in algebraic symbols. One theory that supports students' algebraic thinking abilities is APOS. Learning based on APOS theory is learning with constructivism philosophy that aims to make students build their own knowledge and mathematical concepts through a series of physical and mental activities in the form of action, process, object, and scheme. Through learning based on APOS theory, teachers can describe the flow and way of thinking of students in understanding algebraic problems. Then, from the description obtained can be used to design a model or tool of student evaluation in understanding the problems related to algebra. Thus, the implementation of APOS theory in learning not only illustrates but also improves students' algebraic thinking skills at the level of generational activity, transformational level, and global meta-level.

4. Reference

[1] NTCM 2000 *Principles and standards for school mathematics* Reston: NTCM, Inc

[2] Nurhamsiah H & Ahmad D 2016 Analisis Kesulitan Siswa dalam Mempelajari Bentuk Aljabar Berkaitan dengan Konsep dan Prinsip di SMP *JIPP* **5**No 2

[3] Karuniawati D 2016 Kesulitan Siswa dalam Menyelesaikan Soal Operasi Bentuk Aljabar pada Siswa SMP *Skripsi* Universitas Muhammadiyah Surakarta

[4] Dubinsky E & Wilson R T 2013 High School Students' Understanding of The Function Concept *The Journal of Mathematical Behavior **32*** 83-101

[5] Weller K, Arnon I, and Dubinsky 2011 Preservice Teachers' Understandings of the Relation Between a Fraction or Integer and Its Decimal Expansion: Strength and Stability of Belief *Canadian Journal Of Science, Mathematics And Technology Education* **11** 129–159

[6] Arnon I, Cottrill J, Dubinsky E, Oktac A, Fuentes S R, Trigueros M, and Weller K 2014 From Piaget's Theory to APOS Theory: Reflective Abstraction in Learning Mathematics and the Historical Development of APOS Theory In *"APOS Theory: A Framework of research and curriculum Development in Mathematics Education* London: Springer

[7] Piaget J 1970 *Genetic Epistemology* New York: Columbia University Press

[8] Mulyono 2011 Teori APOS dan Implementasinya dalam Pembelajaran *JMEE* **1**(1)

[9] Dubinsky E 2000 Using A Theory of Learning in College Mathematics Courses *MSOR Connection*, **1**(2)

[10] Dubinsky E, Weller K, McDonald M A, and Brown A 2005 Some Historical Issues and Paradoxes regarding the Concept of Infinity: An Apos-Based Analysis: Part 1. *Educational Studies in Mathematics* **58** 335

[11] Asiala M, Brown A, DeVries D J, Dubinsky E, Mathews D, and Thomas K 1997 A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. *Research in Collegiate Mathematics Education* **2** 1-23

[12] Seeley C 2004 A journey in algebraic thinking *Retrieved from http://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Cathy-Seeley/A-Journey-in-Algebraic-Thinking/ on 1 Oktober 2017*

[13] Becker J R and F D Rivera 2007 Generalization in algebra: The foundation of algebraic thinking and reasoning across grades *ZDM Mathematics Education **2008*** 1

[14] Vance J 1998 Number Operations From An Algebraic Perspective *Teaching Children Mathematics* **4** 282-285

[15] Yuniati 2015 *Meningkatkan Kemampuan Berpikir Aljabar, Berpikir Kritis Matematika, dan Self-Regulated Siswa SMP melalui Pembelajaran Core* Bandung: Universitas Pendidikan Indonesia

[16] Hidayanto E 2013 Proses Berpikir Aritmatika dan Berpikir Aljabar Siswa dalam Menyelesaikan Soal Cerita *Prosiding Seminar Nasional Aljabar dan Pembelajarannya*

[17] Kieran C 2004 Algebraic Thinking in the Early Grades: What Is It? *The Mathematics Educator* **8** 139-151

[18] Usiskin F 2013 What does it mean to understand some mathematics? *12th International Congress on Mathematical Education* 1-20

[19] Van de Walle J H, Karp K S, and Bay-Williams J M 2010 *Elementary and middle school mathematics teaching develop-mentally (7th ed)* Boston: Allyn & Bacon

[20] Arnawa I M 2007 Mengembangkan kemampuan mahasiswa dalam memvalidasi bukti pada aljabar abstrak melalui pembelajaran berdasarkan teori APOS *Jurnal Matematika dan Sains* **14**(2)

[21] Selden A and J Selden 2003 Validations of proof considered as texts: can undergraduates tell whether an argument proves a theorem?,*Journal for Research in Mathematics Education* **34** 4-36

[22] Nurrohma N W 2013 *Implementasi Teori Belajar Apos dengan Menggunakan Pendekatan Siklus ACE untuk Meningkatkan Pemahaman Konsep* Universitas Muhammadiya Surakarta Skripsi Tidak Diterbitkan

[23] Zahid M Z, Sujadi I, and Sari D R 2014 Eksplorasi Konstruksi Pengetahuan Matematika Siswa Kelas VIII SMP Negeri 1 Surakarta Menggunakan Teori Action, Process, Object, Scheme (APOS) pada Materi Pokok Faktorisasi Bentuk Aljabar *Jurnal Elektronik Pembelajaran Matematika* **2** 714-726

[24] Inganah S 2015 *Proses Berpikir Aljabar Symbolic Hight Jumping dalam Menggeneralisasikan Pola melalui Pendelatan Semiotik* Universitas Negeri Malang Tesis Tidak Diterbitkan