A Dual Reciprocity Boundary Element Method for Water Infiltration Problems in a Single Flat and Trapezoidal Irrigation Channels

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Abstract. Governing equation of the problem involving infiltration in homogeneous soils is Richard's Equation. This equation can be studied more conveniently by transforming the equation to a modified Helmholtz equation. In this study, a dual-reciprocity boundary element method (DRBEM) is employed to solve the modified Helmholtz equation numerically. Using the solutions obtained, numerical values of the matric flux potensial are then computed. The proposed method is tested on problems involving infiltration from a single flat and trapezoidal irrigation channels in a homogeneous soil. The numerical solutions of both channels are then compared.

1. Introduction

Studies of infiltration problems in homogeneous soils have been considered by numerous researchers, for instance Azis et al. [3], Batu [2], Clements et al. [4], Lobo et al. [6], and Solekhudin [8]. These problems were solved analytically only for very simple cases, such as those considered by Batu. For more realistic, and hence more complicated cases, numerical methods may be needed. Moreover, although analytical methods may be regarded as successful to a certain extent, it is still useful to develop alternative numerical approaches. One approach worth considering is the dual-reciprocity boundary element method (DRBEM), which is known for its flexibility and accuracy in dealing with boundary conditions in many problems.

In this paper, we study problem involving time independent water flow in unsaturated soils from a trapezoidal channel. To solve the problem, the governing equation, a Richards equation, is transformed into a modified Helmholtz equation. To do so, a set of transformations is employed. The modified Helmholtz equation is then solved numerically using a DRBEM. An example of infiltration from single trapezoidal channel is considered to test the method. The solutions obtained are then compared to those obtained from single flat channel.

2. Problem Formulation

In the study of steady infiltration problems, the governing equation that is often used is

$$\frac{\partial}{\partial x}\left(K(\psi)\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial z}\left(K(\psi)\frac{\partial\psi}{\partial z}\right)=\frac{\partial K(\psi)}{\partial z} \quad (1)$$

where K is the hydraulic conductivity and Z is the vertical physical space coordinate, pointing positively downward [7, 10, 11]. Equation (1) is called the Richard's equation, and represents the movement of water in unsaturated soil in two dimensions.

Using the Matric Flux Potential (MFP) [5] defined as

$$\Theta = \int_{-\infty}^{\psi} K(s)ds \tag{2}$$

and an exponential relation

$$K(\psi) = K_0 e^{\alpha\psi}, \quad \alpha > 0 \tag{3}$$

where α is an empirical parameter (L^{-1}), equation (1) can be written as

$$\frac{\partial^2\Theta}{\partial X^2} + \frac{\partial^2\Theta}{\partial Z^2} = \alpha\frac{\partial\Theta}{\partial Z} \tag{4}$$

The horizontal and vertical components of the flux in terms of the MFP are

$$U = -\frac{\partial\Theta}{\partial X} \tag{5}$$

$$V = \alpha\Theta - \frac{\partial\Theta}{\partial Z} \tag{6}$$

respectively.

The flux normal to the surface with outward pointing normal $\mathbf{n} = (n_x, n_z)$ is given by

$$F = -\frac{\partial\Theta}{\partial X}n_X + \left(\alpha\Theta - \frac{\partial\Theta}{\partial Z}\right)n_Z \tag{7}$$

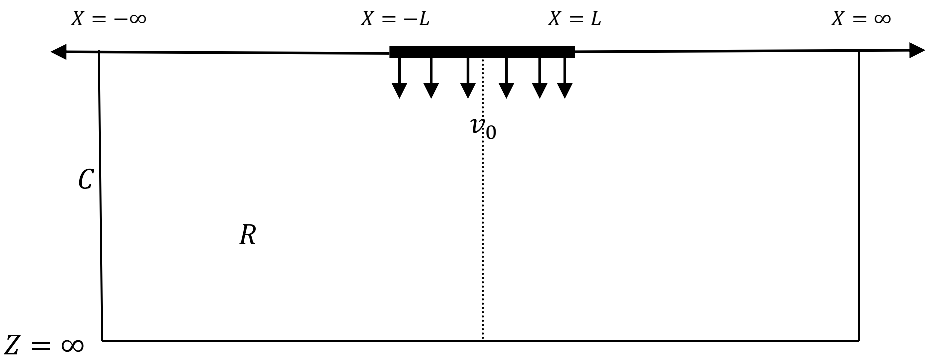

Figure 1

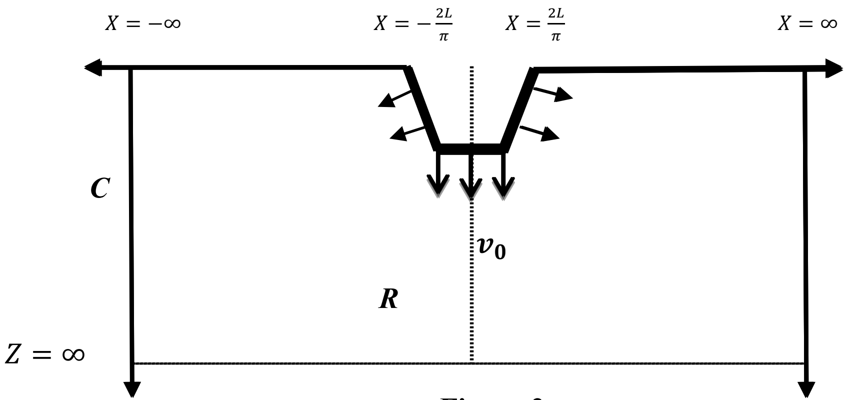

Figure 2

We consider two irrigation channels, with surface length $2L$ and surface length of the soil outside the channelis ∞ as can be seen in Figure 1 and Figure 2. The fluxes on the channel are v_0, and the fluxes on the soil surface outside the channel are 0. The boundary conditions for this problem are [5]

$$F = -v_0, \text{ on the surface of the channel} \quad (8)$$

$$F = 0, \text{ on the surface of the soil outside the channel} \quad (9)$$

$$\Theta = \frac{\partial \Theta}{\partial X} = \frac{\partial \Theta}{\partial Z} = 0,\ X = -\infty \text{ and } Z \geq 0 \quad (10)$$

$$\Theta = \frac{\partial \Theta}{\partial X} = \frac{\partial \Theta}{\partial Z} = 0,\ X = \infty \text{ and } Z \geq 0 \quad (11)$$

and

$$\Theta = \frac{\partial \Theta}{\partial X} = \frac{\partial \Theta}{\partial Z} = 0,\ Z = \infty \text{ and } -\infty \leq X \leq \infty \quad (12)$$

Using dimensionless variables

$$x = \frac{\alpha}{2}X;\quad z = \frac{\alpha}{2}Z;\ \ \Phi = \frac{\pi\Theta}{v_0 L}\ ;\quad u = \frac{2\pi}{v_0\alpha L}U;\quad v = \frac{2\pi}{v_0\alpha L}V;\quad f = \frac{2\pi}{v_0\alpha L}F \quad (13)$$

and the transformation

$$\Phi = \phi e^{z} \quad (14)$$

equation (4) may be written as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \phi \quad (15)$$

Boundary conditions (8) to (12) can be written as

$$\frac{\partial \phi}{\partial n} = \frac{2\pi}{\alpha L}e^{-z} + n_z\phi, \text{ on the surface of the channel} \quad (16)$$

$$\frac{\partial \phi}{\partial n} = -\phi, \text{ on the surface of the soil outside the channel} \quad (17)$$

$$\phi = 0; \frac{\partial \phi}{\partial n} = 0,\ x = -\infty \text{ and } z \geq 0 \quad (18)$$

$$\phi = 0; \frac{\partial \phi}{\partial n} = 0,\ x = \infty \text{ and } z \geq 0 \quad (19)$$

and

$$\phi = 0; \frac{\partial \phi}{\partial n} = 0,\ z = \infty \text{ and } -\infty \leq x \leq \infty \quad (20)$$

Here, $\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x}n_x + \frac{\partial \phi}{\partial z}n_z$ is the normal derivative of ϕ

3. Basic Equation

According to Ang [1], an integral equation to solve equation (15) is

$$\lambda(\xi,\eta)\phi(\xi,\eta) = \int_C \left\{\phi(x,z)\frac{\partial}{\partial n}[\varphi(x,z;\xi,\eta)] - \varphi(x,z;\xi,\eta)\frac{\partial}{\partial n}[\phi(x,z)]\right\} ds(x,z)$$

$$+ \iint_R \varphi(x,z;\xi,\eta)\phi(x,z)dx\, dz \quad (21)$$

where

$$\lambda(\xi,\eta) = \begin{cases} \frac{1}{2}, & (\xi,\eta) \text{ lies on a smooth part of } C \\ 1, & (\xi,\eta) \in R \end{cases} \tag{22}$$

and

$$\varphi(x,z;\xi,\eta) = \frac{1}{4\pi}\ln((x-\xi)^2 + (z-\eta)^2) \tag{23}$$

is the fundamental solution of the Laplace's equation.

Equation (21) may not be solved analytically, and hence a numerical method is needed to solve the integral equation approximately. In this paper, we employ the DRBEM.

4. Result and Discussion

In this section, the values of dimensionless matric flux potential Φ from single trapezoidal channel and single flat channel are presented. The values of Φ are plotted against the dimensionless depth of soil z.

The dimensionless value of the channel width depends upon the value of α which indicate the coarseness or the fineness of a soil. In these DRBEM, Pima Clay Loam is selected as type of soil observed with the value of α is $0.014\ \text{cm}^{-1}$. Furthermore, the reference length L is chosen to be 50π cm that follows $\alpha L = 0.7\pi$ which is the surface length of flat and trapezoidal channel. Using the value of L above, the width and the depth of trapezoidal channel are 200 cm and 75 cm, respectively.

Implementing the DRBEM requires the solution domain to be bounded by a simple closed curve [9]. In this case, we fix the observed soil depth and width are $z = c$ and $x = -b$ to $x = b$, respectively. Following several computational experiments, it was found that setting $c = 4$ and $b = 10$ are sufficient for conditions to be applied at this boundary without significant impact on the value of Φ in the domain.

Table 1: Comparison the values of Φ between trapezoidal channel and flat channel along $x = 0.25$

(*X*, *Z*)	**TRAPEZOIDAL CHANNEL**	**FLAT CHANNEL**
(0.25, 0.55)	0.508581	0.464796
(0.25, 0.60)	0.500177	0.450854
(0.25, 0.65)	0.491193	0.437853
(0.25, 0.70)	0.482230	0.425720
(0.25, 0.75)	0.473427	0.414384
(0.25, 0.80)	0.464782	0.403778
(0.25, 0.85)	0.456257	0.393840
(0.25, 0.90)	0.447842	0.384514
(0.25, 0.95)	0.439558	0.375746
(0.25, 1.00)	0.431441	0.367489
(0.25, 1.50)	0.362635	0.305440
(0.25, 2.00)	0.316786	0.265977
(0.25, 2.50)	0.289037	0.238323
(0.25, 3.00)	0.278746	0.217648
(0.25, 3.50)	0.298053	0.201472

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Table 2: Comparison the values of Φ between trapezoidal channel and flat channel along $x = 0.75$

(*X*, *Z*)	**TRAPEZOIDAL CHANNEL**	**FLAT CHANNEL**
(0.75, 0.05)	0.187154	0.184631
(0.75, 0.10)	0.197485	0.186396
(0.75, 0.15)	0.209573	0.191927
(0.75, 0.20)	0.219985	0.199298
(0.75, 0.25)	0.229897	0.207297
(0.75, 0.30)	0.239345	0.215188
(0.75, 0.35)	0.247479	0.222547
(0.75, 0.40)	0.254846	0.229149
(0.75, 0.45)	0.262503	0.234900
(0.75, 0.50)	0.270403	0.239785
(0.75, 0.55)	0.277611	0.243834
(0.75, 0.60)	0.283195	0.247107
(0.75, 0.65)	0.286789	0.249674
(0.75, 0.70)	0.288597	0.251609
(0.75, 0.75)	0.289073	0.252987
(0.75, 0.80)	0.288670	0.253878
(0.75, 0.85)	0.287737	0.254345
(0.75, 0.90)	0.286521	0.254448
(0.75, 0.95)	0.285182	0.254238
(0.75, 1.00)	0.283819	0.253762
(0.75, 1.50)	0.272792	0.241063
(0.75, 2.00)	0.262009	0.224477
(0.75, 2.50)	0.252319	0.209127
(0.75, 3.00)	0.251556	0.195826
(0.75, 3.50)	0.275733	0.184433

Table 3: Comparison the values of Φ between trapezoidal channel and flat channel along $x = 1$

(*X*, *Z*)	**TRAPEZOIDAL CHANNEL**	**FLAT CHANNEL**
(1.00, 0.05)	0.132416	0.100159
(1.00, 0.10)	0.139676	0.108539
(1.00, 0.15)	0.146416	0.116398
(1.00, 0.20)	0.152790	0.123926
(1.00, 0.25)	0.158914	0.131178
(1.00, 0.30)	0.164839	0.138145
(1.00, 0.35)	0.170582	0.144789
(1.00, 0.40)	0.176146	0.151070
(1.00, 0.45)	0.181532	0.156952

(1.00, 0.50)	0.186742	0.162412
(1.00, 0.55)	0.191770	0.167434
(1.00, 0.60)	0.196601	0.172017
(1.00, 0.65)	0.201208	0.176166
(1.00, 0.70)	0.205553	0.179893
(1.00, 0.75)	0.209601	0.183216
(1.00, 0.80)	0.213328	0.186156
(1.00, 0.85)	0.216728	0.188737
(1.00, 0.90)	0.219810	0.190984
(1.00, 0.95)	0.222595	0.192921
(1.00, 1.00)	0.225110	0.194572
(1.00, 1.50)	0.241951	0.200030
(1.00, 2.00)	0.259710	0.195252
(1.00, 2.50)	0.297009	0.187383
(1.00, 3.00)	0.377588	0.178999
(1.00, 3.50)	0.544425	0.170984

5. CONCLUDING REMARKS

A DRBEM has been formulated and successfully implemented for solving steady infiltration from single flat and trapezoidal channel. The results indicate that the value of dimensionless matric flux potential of trapezoidal channel is higher than these of flat channel.

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