Optimal Control Model of Malaria Spread in South Kalimantan

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Abstract. South Kalimantan is one of the provinces vulnerable to malaria because their work is in the vicinity of forests such as miners, gold miners, forest product seekers even living on the edge of the forests based. Malaria eradication has always been carried out by the government through the South Kalimantan and its achievements have increased, but it is still a problem and needs tougher efforts to achieve malaria free. One way to eradicate the disease is to control it through mathematical modelling SEIR model with infectious force in latent, infected and immune periode. This research analyzes equilibrium point from the malaria distribution model, conducts Optimal Control to complete Mathematical Model of Malaria Spread in South Kalimantan, and accordingly obtains the solution control $u_1(t)$;the vaccination effort given to reduce the number of susceptible individuals into latent individuals and $u_2(t)$; the control of the treatment given to the infected individual to be cured. In the end of this research as a result the solutions will be found in form of controlu_1^* and u_2^*.

1. Introduction

Malaria is a public health problem in Indonesia, including Kalimantan, particularly South Kalimantan. According to Chief Executive Officer by Daily duty executive Head of Provincial Health Office of South Kalimantan Muslim, A total of 96 villages in South Kalimantan are categorized as red or high case of malaria. The villages are spread over nine districts. The highest area is in Tanah Bumbu city with 29 villages. After that, followed by Tabalong Regency with 27 villages. While seven other areas of distribution quite evenly. There are five villages in Banjar Regency, Tanah Laut two villages, Tapin four villages, Hulu Sungai Tengah six villages, Barito Kuala two villages, Kotabaru seven villages, and Balangan fourteen villages. The villages that are still high in malaria cases are mostly from the Meratus Mountains area from Kotabaru to Tabalong Regency.

Malaria is one of the tropical diseases caused by the parasitic infection of Plasmodium that attacks the erythrocytes. Malaria may be acute or chronic. Malaria is caused by anopheles mosquito carrying plasmodium that attacks erotrocytes and is characterized by the discovery of asexual forms in the blood. Malaria can take place without complications or with complications commonly referred to as severe malaria. Symptoms that occur after infection include fever, chills, sweating, headache, nausea, and even vomiting based.

Malaria is transmitted to humans through the bite of Anopheles female mosquito by parasites of the Plasmodium species. After biting the human parasite changes through a complex life cycle. Parasites multiply in the liver and bloodstream of humans. After 10 to 15 days mosquitoes carry parasites and

can infect new people. According to [1], he founds malaria is caused by Plasmodium ovale reported from Flores.

Various problems that involve many systems theory, optimal control and some applications. One of them is the Mathematical Model problem. One is problem about Mathematical Model Malaria. In a previous study "Optimal Control on Determination of Time Interval and Optimal Dosage of Malaria" [2] discussed the formation of malaria disease model then optimize the Determination of Time Interval and Optimal Dose of Malaria Disease. In the following study 'Optimal Control Mathematical Model of Malaria Spread in South Kalimantan' will be discussed modification of malaria spread model SEIR then model used to analyze the rate of invasion of malaria disease based on data obtained sourced from South Kalimantan Provincial Health Office will then be determined model solution by using control theory.

2. Materials and Methods

2.1 Nonhomogen Linear Differential Equation and Solution

Definition 2.1.1 Differential equations are equations containing derivative so one or more dependent variables for one or more independent variables.

Definition 2.1.2 Order linear differential equations-n, with the dependent variable y, and the independent variable x, can be expressed as follows

$$a_0 x \frac{d^n y}{dx^n} + a_1 x \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n(x) y = F(x)$$

With a_0 not equal to zero. If F is equal to zero then the equation reduces to

$$a_0 x \frac{d^n y}{dx^n} + a_1 x \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n xy = 0$$

Called homogeneous differential equation. For F(x) ≠0, referred to as non homogeneous differential equation. [3], [10].

2.2 Basic Reproduction Numbers

The basic reproduction number serves to give information the spread of disease and can be a parameter in providing strategies for disease control. Biologically the reproduction number can mean the average number of secondary infection cases that occur when the infected individual enters the population that are all Susceptible.

Basic reproduction numbers are generally written with R_0, the form R_0 can be one of three possible values:

$R_0 < 1$: The disease will disappear over time
$R_0 = 1$: The disease will become spread and remain on a large scaleepidemic
$R_0 > 1$: There will be an epidemic with a very high degree of relevance with death. [4].

To test the stability properties required calculations to determine the eigen values of the Jacobian matrix at the equilibrium point. As an alternative to determining the eigen values the Ruth – Hurwitz [5], [12].

2.3 Optimal Control

Definition 2.1.3 *Optimal control problem for the system with the target S, the objective function* $J(x_0, t_0, u)$*, the set of admissible control U, and the initial state* x_0 *at time* t_0*isdecisivecontrolu* ϵ *U that maximizes objective function* $J(u)$*. Any control u* which provides a solution to the problem of optimal control called optimal control.*

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In the following discussion, the problems given in the case of optimal control with state end and the end time is known. In other words, the target set S shaped S = $\{x_1\}$x$\{t_1\}$ in the form (x_1,t_1) with x_1 specially element in R^n and t_1 elementat (T_1,T_2).
Given the state system by the end and the end time unknown

$$\dot{x}(t) = f[x(t), u(t), t]$$

with x(t) vector state sized$nx1$, u(t) input vector sized$mx1$, f a vector valued function. Initially given state is X_0 and initially time ist$_0$. Target set S form (x_1,t_1)witht$_1$ ϵ (T_1, T_2) known value and $t_1 > t_0$ .Optimal control problem is to find the admissible control u(t) with the initial value $(x_0,t_0)$and thefinal value (x_1,t_1) thatmaximizesthe objective function$J(u) = \int_{t_0}^{t_1} L(x(t), u(t), t)\ dt$. To solve the problems mentioned above optimal control, first determined necessary condition for optimal control are met.[6], [13].
To solve the above-mentioned optimal control problem, a prerequisite is necessary to ensure optimal control is fulfilled. This requirement is derived using the Lagrange approach as follows:

1. Given a maximizing problem J (u) with a particular state, suppose the initial state is x(t_0) = X_0 and the final state let x(t_1) = x_1.
2. Scalar function is defined by £ ≡£ $[x(t), u(t), \lambda(t), t, \mu]$ with
   $£ = \int_{t_0}^{t_1} L(x(t),u(t),t)\ dt - \int_{t_0}^{t_1} \lambda(t)\big(\dot{x}(t) - f\big(x(t)\big), u(t), t\big)dt - \mu(x(t_0) - (x_0))\ dt.$
3. Defined scalar function $\mathrm{H}(\mathrm{x}, \mathrm{u}, \lambda, t)$with
   $\mathrm{H}(\mathrm{x}(t), \mathrm{u}(t), \lambda(t), t) = [\mathrm{L}(\mathrm{x}(t), \mathrm{u}(t), t) + \lambda(t)\big(f\big(x(t)\big), u(t), t\big)]$ the function H is called the Hamiltonian function (briefly called Hamiltonian) of the system. From the definition of this H function Lagrangian can be written into the Hamiltonian function. The process is as follows:
   $£ = \int_{t_0}^{t_1} L(x(t),u(t),t)\ dt - \int_{t_0}^{t_1} \lambda(t)\big(\dot{x}(t) - f\big(x(t)\big), u(t), t\big)dt - \mu(x(t_0) - (x_0))\ dt$

Then both sides are integrated from to obtain Hamiltonian function

$$£ = \int_{t_0}^{t_1} [H(x(t),u(t),\lambda(t),t) + \dot{\lambda}(t)x(t)]dt - \lambda(t_1)x(t_1) + \lambda(t_0)x(t_0)$$

$$-\mu\big(x(t_0) - (x_0)\big).$$

It is assumed that at the optimal time, control variables and state variables $u^*(t)$ and $x^*(t)$

$$£ = \int_{t_0}^{t_1} [H(x(t,\varepsilon),u(t,\varepsilon),\lambda(t),t) + \dot{\lambda}(t)x(t,\varepsilon)]dt - \lambda(t_1)x(t_1,\varepsilon) + \lambda(t_0)x(t_0)$$

$$-\mu\big(x(t_0) - (x_0)\big)$$

with$£ = £(x(t,\varepsilon), u(t,\varepsilon), \lambda(t), t, \mu)$.In the previous process, if in fact optimally, then the deviation is so small that it does not affect the value of the Lagrange function

$$\frac{\partial £(x(t,\varepsilon),u(t,\varepsilon),\lambda(t),t,\mu)}{\partial \varepsilon} = 0.$$

This result is further used to calculate the requirement of maximum principle. When evaluated at the optimal value of the control variable and the state variable, includes several components. The partial derivative will be equal to zero if each component is zero, can be explained as follows.The left-hand equation can be written

$$\frac{\partial £}{\partial \varepsilon} = \frac{\partial\left[\int_{t_0}^{t_1} \mathrm{H}(x(t,\varepsilon),u(t,\varepsilon),\lambda(t),t)+\lambda(t)x(t,\varepsilon)\right]\mathrm{dt}}{\partial \varepsilon} - \frac{\partial\lambda(t_1)x(t_1,\varepsilon)}{\partial \varepsilon} + \frac{\partial\lambda(t_0)x(t_0)}{\partial \varepsilon}\ \frac{-\partial\mu\big(x(t_0)-(x_0)\big)}{\partial \varepsilon}$$

$\mathrm{H}^* \equiv \mathrm{H}(x(t,\varepsilon), u(t,\varepsilon), \lambda(t), t)$with$x(t,\varepsilon), u(t,\varepsilon)$.

Because of $\frac{\partial £}{\partial \varepsilon} = 0$ result as

$$\int_{t_0}^{t_1} \left[\frac{\partial \mathrm{H}^*}{\partial u} P_1(\mathrm{t}) + \left[\frac{\partial \mathrm{H}^*}{\partial x} + \dot{\lambda}(t)\right] P_2(\mathrm{t})\right] \mathrm{dt} - \lambda(t_1)dx(t_1) = 0.$$

From the whole process and some of the conditions mentioned above, the following requirements. If $u^*(t)$ and $x^*(t)$maximize$J(u) = \int_{t_0}^{t_1}[L\ (x(t),u(t),t)\ dt$ with state equations$\dot{x}(t) = f\ ((x(t),u(t),t)$
for the Hamiltonian function $H(x(t),u(t),\lambda(t),t) = L\ (x(t),u(t),t) + \lambda(t)f\ (x(t),u(t),t)$.
Optimal value $u^*(t)$and $x^*(t)$ should be

- $H(x^*(t),u^*(t),\lambda(t),t) \geq H(x^*(t),u(t),\lambda(t),t),\forall\ t.$
- $\frac{\partial H(x^*(t),u^*(t),\lambda(t),t)}{\partial xt} = -\dot{\lambda}(t).$
- $\frac{\partial H(x^*(t),u^*(t),\lambda(t),t)}{\partial \lambda t} = \dot{x}(t).$
- $x(t_0) = x_0$.This condition is called the Pontryagin Maximum Principle.[7],[11]

3. Discussion of problems

Patients with malaria with gametocytes become a source of transmission with an intermediary is a mosquito as a vector. Humans are an important reservoir, but the disease is transmitted through the bite of a female anopheles mosquito containing the plasmodium species parasite. Most occur naturally, through the bite of a female anopheles mosquito, rarely transmitted by blood transfusion and / or bone marrow transplantation.

When female anopheles mosquitoes suck blood containing gametocytes, then in the body of the mosquito occurs fertilization that produces a zygote. The zygote develops into an ookinet then penetrates the gastric wall of the mosquito. On the outer wall of the ookinetic mosquito will become an oxista and then become sporozoit that is infective and ready to be transmitted to humans. When a mosquito sucks human blood, it will enter the bloodstream. Approximately 9 to 14 days after humans are bitten, the symptoms of malaria appear.

Malaria is one of the epidemic diseases and can be model in mathematical models. The establishment of this malaria dispersion model follows a SEIR model consisting of four compartments:

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Malaria is one of the epidemic diseases and can be modeled in mathematical models. The establishment of this malaria distribution model follows the SEIR [8],[9] model consisting of four compartments:

a. Class S (Susceptible) declares a vulnerable individual class
b. Class E (Exposed) states the class of individuals who have been infected but not sick (latent)
c. Class I (Infectious) states the class of individuals who have contracted the disease
d. Class R (Recovered) states the class of individuals who have recovered from the disease.

The assumptions used in the malaria distribution model are as follows:
Populations are assumed to be open so that there is an increasing (incoming) or decreasing (out) population of the population. Individuals who are in latent class, infection and heal have the possibility to transmit the disease, the disease causes death (fatal), the incubation period is long enough. Natural birth and death are constant, There is only malaria disease in the population, The disease spreads

through contact between individuals with mosquitoes, Every individual born directly into the susceptible group, Malaria-infected individuals can recover from illness and can experience death only due to illness and individual infected malaria can heal because of short cycles of malaria and the presence of natural immunity. Based on the above problems can be made the following transfer diagram

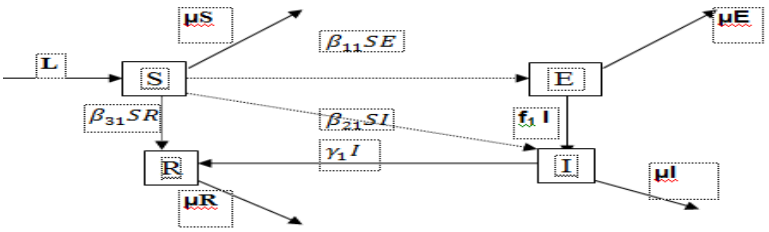

Figure.1 Diagram Model SEIR of Malaria spread

Based on the above transfer diagram can be formulated model as follows :

$\frac{dS}{dt} = L - \beta_{11}SE - \beta_{21}SI - \beta_{31}SR - \mu S$ (3.1)

$\frac{dE}{dt} = \beta_{11}SE + \beta_{21}SI + \beta_{31}SR - \gamma_1 E - \mu E$ (3.2)

$\frac{dI}{dt} = \gamma_1 E - f_1 I - \mu I$ (3.3)

$\frac{dR}{dt} = f_1 I - \mu R$ (3.4)

The definition of the parameters used in the model is

L denotes the constant rate of birth rate

β_{11}denotes the rate of effective contact from vulnerable to latent

β_{21}denotes the rate of effective contact from vulnerable to becoming infected

β_{31}states the rate of effective contact from vulnerable to healed

μstates the natural rate of death

f denotes the rate from the latent class to the infection class

$\gamma_1$denotes the rate of infection class to be healed

Given N (t) denotes the population size at time t, then the equation of

$N(t) = S(t) + E(t) + I(t) + R(t)$.

To simplify System (3.1) to (3.4) for example μ dt = dτ or τ = μt, then System (3.1) to (3.4) becomes

$\frac{dS}{d\tau} = \frac{L}{\mu} - \beta_1 SE - \beta_2 SI - \beta_3 SR - S$ (3.6)

$\frac{dE}{d\tau} = \beta_1 SE + \beta_2 SI + \beta_3 SR - (1+\gamma)E$ (3.7)

$\frac{dI}{d\tau} = \gamma E - (1+f)I$ (3.8)

$\frac{dR}{d\tau} = fI - R$ (3.9)

with

$\beta_1 = \frac{\beta_{11}}{\mu}, \beta_2 = \frac{\beta_{21}}{\mu}, \beta_3 = \frac{\beta_{31}}{\mu}, \gamma = \frac{\gamma_1}{\mu}, \ f = \frac{f_1}{\mu}$.For system (3.6) to (3.9) becomes $\frac{dN}{d\tau} = \frac{dS}{d\tau} + \frac{dE}{d\tau} + \frac{dI}{d\tau} + \frac{dR}{d\tau}$

3.1 The point of equilibrium

The point of equilibrium System (3.6) to (3.9) equation becomes

$\frac{L}{\mu} - \beta_1 SE - \beta_2 SI - \beta_3 SR - S = 0$,(3.10)

$\beta_1 SE + \beta_2 SI + \beta_3 SR - \delta E = 0$, (3.11)

$$\gamma E - \omega I = 0, \quad (3.12)$$
$$kI - R = 0, \quad (3.13)$$
$$\frac{L}{\mu} - N - E - I = 0. \quad (3.14)$$

If $I = 0$, equation becomes (3.12) becomes $\gamma E = 0$ and $\Leftrightarrow E = 0$.From equations (3.13) value $R = 0$, and equation (3.10) becomes $\frac{L}{\mu} - S = 0 \Leftrightarrow S = \frac{L}{\mu}$.Thus, the free equilibrium point $P_0(S, E, I, R) = \left(\frac{L}{\mu}, 0,0,0\right)$. Next will be sought an endemic equilibrium point that is if $I > 0$.

From Equation (3.12) is obtained$I = \frac{\gamma E}{\omega}$ (3.15)

Equation(3.14) is obtained $E = \frac{L}{\mu} - N - I$. (3.16)

Next from (3.15) and (3.16) have $I = \frac{\gamma\left[\frac{L}{\mu} - N - I\right]}{\omega}$, so from (3.12) obtain $E = \frac{\omega[L-\mu N]}{\mu(\omega+\gamma)}$. From (3.13) obtain $R = \frac{f\gamma[L-\mu N]}{\mu(\omega+\gamma)}$. From (3.10) $\frac{L}{\mu} = (\beta_1 E + \beta_2 I + \beta_3 R + 1)S$

So $S = \frac{L(\omega+\gamma)}{(\beta_1\omega+\beta_2\gamma+\beta_3 f\gamma)(L-\mu N)+\mu(\omega+\gamma)}$(3.17)

$$E = \frac{\omega[L-\mu N]}{\mu(\omega+\gamma)} \quad (3.18)$$
$$I = \frac{\gamma[L-\mu N]}{\mu(\omega+\gamma)} \quad (3.19)$$
$$R = \frac{k\gamma[L-\mu N]}{\mu(\omega+\gamma)}. \quad (3.20)$$

If system From (3.17) combination to (3.20) result

$$\Leftrightarrow \left(\frac{\beta_1\omega[L-\mu N]}{\mu(\omega+\gamma)} + \frac{\beta_2\gamma[L-\mu N]}{\omega(\mu)+\gamma\mu} + \frac{\beta_3 f\gamma[L-\mu N]}{\mu(\omega+\gamma)}\right)\left(N - \frac{\omega[L-\mu N]}{\mu(\omega+\gamma)} - \frac{\gamma[L-\mu N]}{\omega(\mu)+\gamma\mu} - \frac{f\gamma[L-\mu N]}{\mu(\omega+\gamma)}\right)$$
$$- \frac{\delta\omega[L-\mu N]}{\mu(\omega+\gamma)} = 0$$
$$\Leftrightarrow \left[\left(\mu N\delta\omega - L(\delta\omega - \omega - \gamma)\right)(\beta_1\omega + \beta_2\gamma + \beta_3 f\gamma) - \delta\omega\mu(\omega+\gamma)\right][L-\mu N] = 0$$
$$\Leftrightarrow F(N)(L-\mu N) = 0, \quad (3.21)$$

With $F(N) = \left(\mu N\delta\omega - A(\delta\omega - \alpha_1\omega - \alpha_2\gamma)\right)(\beta_1\omega + \beta_2\gamma + \beta_3 k\gamma) - \delta\omega\mu(\alpha_1\omega + \alpha_2\gamma)$.

For value $(L - \mu N) > 0$, equations (3.21)must $F(N) = 0$. For $N = 0$obtain

$$F(0) = -L(\delta\omega - \omega - \gamma)(\beta_1\omega + \beta_2\gamma + \beta_3 f\gamma) - \delta\omega\mu(\omega+\gamma) < 0.$$

And for $N = \frac{L}{\mu}$obtain

$$F\left(\frac{L}{\mu}\right) = \mu\delta\omega\left(\frac{L}{\mu}\right) - L(\delta\omega - \omega - \gamma)(\beta_1\omega + \beta_2\gamma + \beta_3 f\gamma) - \delta\omega\mu(\omega+\gamma)$$
$$= \mu(\omega+\gamma)\delta\omega[R_0 - 1],$$

With used $S = \frac{L(\omega+\gamma)}{(\beta_1\omega+\beta_2\gamma+\beta_3 f\gamma)(L-\mu N)+\mu(\omega+\gamma)}$from equations from (3.17)can value

$R_0 = \frac{L(\beta_1\omega+\beta_2\gamma+\beta_3 f\gamma)}{\delta\omega\mu}$and $R_0 = \frac{1}{S}$.

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3.2 Stability of the point of illness free equilibrium

From system (3.14) to (3.17), then those functions are partially derived

$$g_1(S,E,I,R) = \frac{A}{\mu} - \beta_1 SE - \beta_2 SI - \beta_3 SR - S$$
$$g_2(S,E,I,R) = \beta_1 SE + \beta_2 SI + \beta_3 SR - \delta E$$
$$g_3(S,E,I,R) = \gamma E - \omega I$$
$$g_4(S,E,I,R) = kI - R$$

With derivative function to S, E, I, Robtains

$$\frac{\partial g_1}{\partial S} = -\beta_1 E - \beta_2 I - \beta_3 R - 1 \frac{\partial g_1}{\partial E} = -\beta_1 S \frac{\partial g_1}{\partial I} = -\beta_2 S \frac{\partial g_1}{\partial R} = -\beta_3 S.$$
$$\frac{\partial g_2}{\partial S} = \beta_1 E + \beta_2 I + \beta_3 R \frac{\partial g_2}{\partial E} = \beta_1 S - \delta \frac{\partial g_2}{\partial I} = \beta_2 S \frac{\partial g_2}{\partial R} = \beta_3 S.$$
$$\frac{\partial g_3}{\partial S} = 0 \frac{\partial g_3}{\partial E} = \gamma \frac{\partial g_3}{\partial I} = -\omega \frac{\partial g_3}{\partial R} = 0.$$
$$\frac{\partial g_4}{\partial S} = 0 \frac{\partial g_4}{\partial E} = 0 \frac{\partial g_4}{\partial I} = k \frac{\partial g_4}{\partial R} = -1.$$

After the above-mentioned functions are partially derived from S, E, I, R the Jacobian matrix of System (3.6) to (3.9) as follows :

$$J\big(g(S,E,I,R)\big) = \begin{bmatrix} \frac{\partial g_1(S,E,I,R)}{\partial S} & \frac{\partial g_1(S,E,I,R)}{\partial E} & \frac{\partial g_1(S,E,I,R)}{\partial I} & \frac{\partial g_1(S,E,I,R)}{\partial R} \\ \frac{\partial g_2(S,E,I,R)}{\partial S} & \frac{\partial g_2(S,E,I,R)}{\partial E} & \frac{\partial g_2(S,E,I,R)}{\partial I} & \frac{\partial g_2(S,E,I,R)}{\partial R} \\ \frac{\partial g_3(S,E,I,R)}{\partial S} & \frac{\partial g_3(S,E,I,R)}{\partial E} & \frac{\partial g_3(S,E,I,R)}{\partial I} & \frac{\partial g_3(S,E,I,R)}{\partial R} \\ \frac{\partial g_4(S,E,I,R)}{\partial S} & \frac{\partial g_3(S,E,I,R)}{\partial E} & \frac{\partial g_3(S,E,I,R)}{\partial I} & \frac{\partial g_3(S,E,I,R)}{\partial R} \end{bmatrix}$$
$$= \begin{bmatrix} -\beta_1 E - \beta_2 I - \beta_3 R - 1 & -\beta_1 S & -\beta_2 S & -\beta_3 S \\ \beta_1 E + \beta_2 I + \beta_3 R & \beta_1 S - \delta & \beta_2 S & \beta_3 S \\ 0 & \gamma & -\omega & 0 \\ 0 & 0 & f & -1 \end{bmatrix}.$$

To look for the stability of the disease-free equilibrium point can be seen from Jacobian matrix at equilibrium point $P_0\left(\frac{L}{\mu}, 0,0,0\right)$is $Jf(P_0) = J\left(f\left(\frac{L}{\mu}, 0,0,0\right)\right)$

$$= \begin{bmatrix} -1 & \frac{-\beta_1 L}{\mu} & \frac{-\beta_2 L}{\mu} & \frac{-\beta_3 L}{\mu} \\ 0 & \frac{\beta_1 L}{\mu} - \delta & \frac{\beta_2 L}{\mu} & \frac{\beta_3 L}{\mu} \\ 0 & \gamma & -\omega & 0 \\ 0 & 0 & f & -1 \end{bmatrix}.$$

The characteristic polynomial $Jf(P_0)$is $P(\lambda) = \det\big(\lambda I - Jf(P_0)\big) = |\lambda I - Jf(P_0)|$.The characteristic equation is$|\lambda I - Jf(P_0)| = 0$

$$\Leftrightarrow \left| \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & \frac{-\beta_1 L}{\mu} & \frac{-\beta_2 L}{\mu} & \frac{-\beta_3 L}{\mu} \\ 0 & \frac{\beta_1 L}{\mu} - \delta & \frac{\beta_2 L}{\mu} & \frac{\beta_3 L}{\mu} \\ 0 & \gamma & -\omega & 0 \\ 0 & 0 & k & -1 \end{bmatrix} \right| = 0$$

$$(\lambda + 1)\left[\lambda^3 + \left(1 + \delta + \omega - \frac{\beta_1 L}{\mu}\right)\lambda^2 + \left(\delta + \delta\omega + \omega - \frac{\beta_1 L}{\mu} - \frac{\beta_1 L\omega}{\mu} - \frac{\beta_2 L\gamma}{\mu}\right)\lambda + \left(\delta\omega - \frac{\beta_1 L\omega}{\mu} - \frac{\beta_2 L\gamma}{\mu} - \frac{\beta_3 L\gamma f}{\mu}\right)\right] = 0. \quad (3.21)$$

Equation (3.21) can make

$(\lambda + 1)(\lambda^3 + A\lambda^2 + B\lambda + C) = 0$,(3.22)

consequently all the real parts of the eigenvalues on the matrix $Jf(P_0)$are negative for $R_0 < 1$ So the equilibrium point$P_0\left(\frac{A}{\mu}, 0,0,0\right)$is stable asymptotically local.This means that when the number of infected populations is smaller than the number of individuals who recover plus individuals who die both from illness and natural death, the value of eigen is negative.Means in time the system will stabilize toward the point of disease free equilibrium.

3.3 Basic Reproduction Numbers

The basic reproduction number (R_0) is a particular parameter used to determine the dynamics of disease spread, $R_0$is used as a quantity to indicate the extent of disease spread. To analyze the infection rate of malaria, used a parameter that is

$R_0 = \frac{L(\beta_1\omega + \beta_2\gamma + \beta_3 f\gamma)}{\delta\omega\mu}$ and $R_0 = \frac{1}{S_0}$. Obtain two conditions about it, that is as follows :

(i) $\frac{dI}{dt} > 0$there will be an increase in the number of infected populations, and

(ii)$\frac{dI}{dt} < 0$the disease will not spread so the number of infected population will not increase.

Because the desired is when the disease does not spread if $R_0 < \frac{1}{S_0}$. Based on the statement of Driessche & Watmough (2005) the disease will disappear over time if $R_0 < 1$.Next let us say $R_0 = \frac{K}{S_0}$for $1 \geq K \geq 0$.

Table 3.1 The total population of South Kalimantan in 2017

No.	District / City	Population
1.	Tanah Laut	334328
2.	Kotabaru	331326
3.	Banjar	571573
4.	Barito Kuala	306 195
5.	Tapin	186 672
6.	Hulu Sungai Selatan	232587
7.	Hulu Sungai Tengah	266501
8.	Hulu Sungai Utara	231594
9.	Tabalong	247106
10.	Tanah Bumbu	343193

No.	District / City	Population
11.	Balangan	127503
12.	Banjarmasin	692793
13.	Banjarbaru	248423

Source: Central Bureau of Statistics of South Kalimantan Province

Table 3.2 Malaria case in South Kalimantan in 2013-2017

No.	District / City	Case (Year)				
		2013	2014	2015	2016	2017
1	Tanah Laut	450	291	119	74	135
2	Kotabaru	1447	1122	297	174	202
3	Banjar	652	578	179	176	290
4	Barito Kuala	227	88	44	18	201
5	Tapin	406	204	111	101	98
6	Hulu Sungai Selatan	308	213	71	42	148
7	Hulu Sungai Tengah	182	187	114	65	169
8	Hulu Sungai Utara	78	66	44	38	219
9	Tabalong	1139	1711	1711	938	131
10	Tanah Bumbu	1586	451	256	166	147
11	Balangan	119	251	309	299	157
12	Banjarmasin	40	31	7	3	3
13	Banjarbaru	162	74	139	8	20

Source: South Kalimantan Provincial Health Office

So from populations obtainR_0every District / City inSouth Kalimantan province used value K interval $1 \geq K \geq 0$, $K = 0{,}85$ obtains after the calculation

Table 3.3 The calculation results value$R_0$for $K = 0{,}85$

No.	District / City	Value R_0				
		2013	2014	2015	2016	2017
1.	Tanah Laut	0,000275518	0,000271	0,000266	0,000262	0,000258
2.	Kotabaru	0,000280545	0,000275	0,00027	0,000265	0,000261
3.	Banjar	0,000161231	0,000158	0,000156	0,000153	0,000151
5.	Tapin	0,000297624	0,000293	0,000289	0,000285	0,000281
4.	Barito Kuala	0,000488874	0,000482	0,000474	0,000468	0,000461
6.	HSS	0,00038831	0,000384	0,000379	0,000374	0,00037
7.	HST	0,000339044	0,000335	0,000331	0,000327	0,000323
8.	HSU	0,000393555	0,000388	0,000382	0,000377	0,000372

9.	Tabalong	0,000373275	0,000367	0,000361	0,000355	0,000349
10.	Tanah Bumbu	0,000288104	0,000278	0,000269	0,000261	0,000254
11.	Balangan	0,00072595	0,000713	0,000701	0,000689	0,000677
12.	Banjarmasin	0,000131294	0,000129	0,000128	0,000126	0,000124
13.	Banjarbaru	0,000397176	0,000385	0,000374	0,000363	0,000352

Based on table 3.3 with the value of K given the smaller the value of R_0 obtained is also smaller from year to year every Regency/City. This is due to the fact that when the large total population, with small infected populations, has a large vulnerable population, so the value of R_0obtained is small. Whereas if the total population is small, with a large infected population then the small vulnerable population so that the value of R_0 obtained is also large.

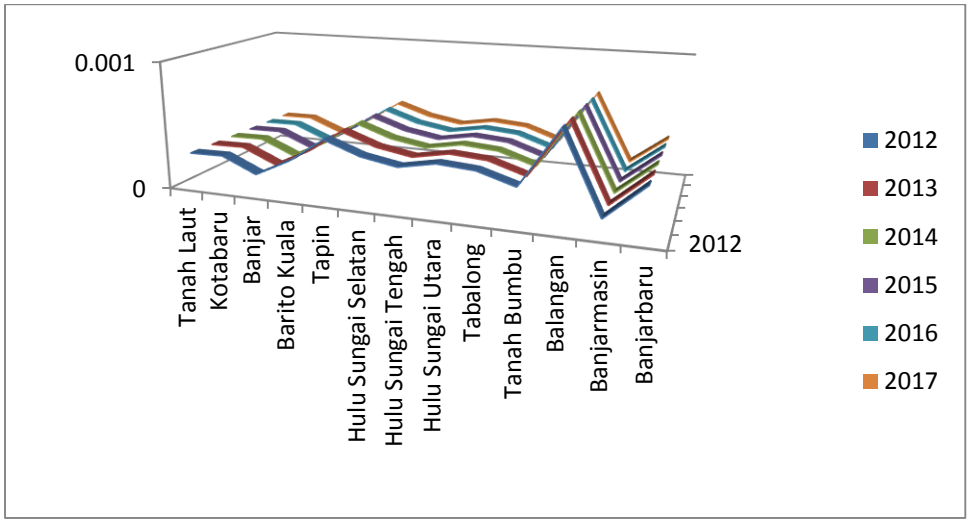

Figure 2. The spread of malaria infection rate for K = 0.85

3.4 Optimal Control for Malaria Distribution Model.

Optimal control of the Malaria distribution model aims to maximize the number of healthy
The state equation is:

$$\dot{x} = \begin{bmatrix} \dot{S} \\ \dot{E} \\ \dot{I} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} \frac{L}{\mu} - \beta_1 SE - \beta_2 SI - \beta_3 SR - S \\ \beta_1 SE + \beta_2 SI + \beta_3 SR - (1+\gamma)E \\ \gamma E - (1+f)I \\ fI - R \end{bmatrix} \dot{x} = \begin{bmatrix} \frac{L}{\mu} - \beta_1 SE - \beta_2 SI - \beta_3 SR - S \\ \beta_1 SE + \beta_2 SI + \beta_3 SR - (1+\gamma)E \\ \gamma E - (1+f)I \\ fI - R \end{bmatrix}$$

Control will be given to minimize the number of infected individual populations.Controls are given control actions $u_1(t)$ and $u_2(t)$. Control $u_1(t)$ is the vaccination effort given to reduce the number of susceptible individuals into latent individuals, $u_2(t)$ represents the control of the treatment given to the infected individual to be cured. So the equation

$$\frac{dS}{d\tau} = \frac{L}{\mu} - (1+u_1)\beta_1 SE - \beta_2 SI - \beta_3 SR - S \qquad (3.6)$$

$$\frac{dE}{d\tau} = (1+u_1)\beta_1 SE + \beta_2 SI + \beta_3 SR - (1+(1+u_1)\gamma)E \quad (3.7)$$

$$\frac{dI}{d\tau} = (1+u_2)\gamma E - (1+f)I \qquad (3.8)$$

$\frac{dR}{d\tau} = fI - R$ (3.9)

With equations *performance index* is:

$$J(u) = \int_{t_0}^{t_1} L(x(t), u(t), t)dt$$

valuet$_0$ adalah first time, t_f is finishing time, x(t) is state variable and u(t) is control variable. Equations performance index expressed as the system from the initial state to the final state is:$J(u) = \int_{t_0}^{t_f}(Ai + C_1{u_1}^2 + C_2{u_2}^2)dt$

Constraint equations :

$\frac{dS}{d\tau} = \frac{L}{\mu} - (1+u_1)\beta_1 SE - \beta_2 SI - \beta_3 SR - S$

$\frac{dE}{d\tau} = (1+u_1)\beta_1 SE + \beta_2 SI + \beta_3 SR - (1+(1+u_2)\gamma)E$

$\frac{dI}{d\tau} = (1+u_2)\gamma E - (1+f)I$

$$\frac{dR}{d\tau} = fI - R$$

$0 \le t \le t_f 0 \le u_1 \le 1 0 \le u_2 \le 1$

Where $t_f$is the end time,coefficient A is the number of infected individuals, C_1represents the vaccination cost given to reduce the number of susceptible individuals into latent individuals, C_2represents the cost of treatment given to the infected individual to be cured. By completing the objective function will be obtained optimal control u1 * and u2 *.

Then, based on the principle of maximum Pontryagin the first step is to determine the Hamiltonian function is as follows:

$H(x(t), u(t), \lambda(t), t) = L(x(t), u(t), t) + \lambda(t), f(x(t), u(t), t)$

$$= \left(Ai + C_1{u_1}^2 + C_2{u_2}^2\right) + (\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4)\begin{bmatrix} \frac{L}{\mu} - (1+u_1)\beta_1 SE - \beta_2 SI - \beta_3 SR - S \\ (1+u_1)\beta_1 SE + \beta_2 SI + \beta_3 SR - (1+(1+u_2)\gamma)E \\ (1+u_2)\gamma E - (1+f)I \\ fI - R \end{bmatrix}$$

$$= \left(Ai + C_1{u_1}^2 + C_2{u_2}^2\right) + \lambda_1(\frac{L}{\mu} - (1+u_1)\beta_1 SE - \beta_2 SI - \beta_3 SR - S) + \lambda_2((1+u_1)\beta_1 SE + \beta_2 SI + \beta_3 SR - (1+(1+u_1)\gamma)E) + \lambda_3\ (1+u_2)\gamma E - (1+f)I + \lambda_4(fI - R)$$

The second step is assumed that at the optimum time, the control variables and state variables are respectively

$u^*(t)$ dan $x^*(t)$must :$\frac{\partial H}{\partial u} = 0$it mean $\frac{\partial H}{\partial u_1} = 0$and $\frac{\partial H}{\partial u_2} = 0, \frac{\partial H}{\partial u_1} = 0$so $(2C_1u_1 - \beta_1 SE + \lambda_2\beta_1 SE) = 0$ and $2C_1u_1 = \beta_1 SE - \lambda_2\beta_1 SE$. So for the equations obtain $u_1 = \frac{\beta_1 SE - \lambda_2\beta_1 SE}{2C_1}$it meanu$_1{}^* = \frac{\beta_1 SE - \lambda_2\beta_1 SE}{2C_1}$. The next step $\frac{\partial H}{\partial u_2} = 02C_2u_2 + \lambda_3\ \gamma E - \gamma E = 0$it meanu$_2{}^* = \frac{\gamma E - \lambda_3\ \gamma E}{2C_2}$. Because valueu$_1{}^*$ is$0 \le u_1{}^* \le 1$,so canu*(t) is $0 \le u_1{}^* \le 1$can to obtain the following possibilities:

$$u_1{}^*[(t)] = \begin{cases} 0\ , & \text{if } \frac{\beta_1 SE - \lambda_2\beta_1 SE}{2C_1} \le 0 \\ \frac{\beta_1 SE - \lambda_2\beta_1 SE}{2C_1}, & \text{if } 0 < \frac{\beta_1 SE - \lambda_2\beta_1 SE}{2C_1} < 1 \\ 1, & \text{if } \frac{\beta_1 SE - \lambda_2\beta_1 SE}{2C_1} \ge 1 \end{cases}$$

And valueu$_2{}^*$ is $0 \le u_2{}^* \le 1$so can u*(t) is $0 \le u_2{}^* \le 1$so as to obtain the following

$$u_2{}^*[(t)] = \begin{cases} 0\ , & \text{if } \dfrac{\gamma E - \lambda_3\,\gamma E}{2C_2} \le 0 \\ \dfrac{\gamma E - \lambda_3\,\gamma E}{2C_2}, & \text{if } 0 < \dfrac{\gamma E - \lambda_3\,\gamma E}{2C_2} < 1 \\ 1, & \text{if } \dfrac{\gamma E - \lambda_3\,\gamma E}{2C_2} \ge 1 \end{cases}$$

Based on the above three possibilities, the first possibility in the calculation of the mathematical value obtained

$u_1{}^*$is$0 < \frac{\beta_1 SE - \lambda_2 \beta_1 SE}{2C_1} < 1$ and $u_2{}^*$ is $0 < \frac{\gamma E - \lambda_3 \gamma E}{2C_2} < 1$ in this case meet the limits$0 \le u^*(t) \le 1$ so the value$u_1{}^*[(t)] = \frac{\beta_1 SE - \lambda_2 \beta_1 SE}{2C_1}$ and $u_2{}^*[(t)] = \frac{\gamma E - \lambda_3\,\gamma E}{2C_2}$

Furthermore, of the three possible values of $u^*(t)$ obtained then a value can be determined$u^*(t)$ which is optimal in step third. Because$u(t) \in U = \{u(t) | 0 \le u(t) \le 1, t_0 < t < t_f\}$so as to obtain the value of the smallest limit (supremum) as follows:

$u_1{}^* = \max\left(0, \frac{\beta_1 SE - \lambda_2 \beta_1 SE}{2C_1}\right)$ and $u_2{}^*[(t)] = \max\left(0, \frac{\gamma E - \lambda_3\,\gamma E}{2C_2}\right)$and obtained the value of the largest lower limit (infimum) as follows:

$u_1{}^* = \min\left(\max\left(0, \frac{\beta_1 SE - \lambda_2 \beta_1 SE}{2C_1}\right), 1\right)$ and $u_2{}^* = \min\left(\max\left(0, \frac{\gamma E - \lambda_3 \gamma E}{2C_2}\right), 1\right)$.

Then, the fourth step is to solve the state equation because of the controller form$u^*(t)$contains state variables(S, E, I, R)that is$\dot{x} = \frac{\partial H}{\partial \lambda}$so as to obtain the optimal state equation as follows:

$$\frac{dS}{d\tau} = \frac{L}{\mu} - (1 + u_1{}^*)\beta_1 SE - \beta_2 SI - \beta_3 SR - S$$

$$\frac{dE}{d\tau} = (1 + u_1{}^*)\beta_1 SE + \beta_2 SI + \beta_3 SR - (1 + (1 + u_2{}^*)\gamma)E$$

$$\frac{dI}{d\tau} = (1 + u_2{}^*)\gamma E - (1 + f)I$$

$$\frac{dR}{d\tau} = fI - R$$

In addition to state variables there are also variable costate $\lambda_1, \lambda_2, \lambda_3$and $\lambda_4$on the form of the controller $u^*(t)$it is necessary to solve the costate equation to obtain the optimal costate equation in the fifth step is as follows:

$\dot{\lambda^*} = \frac{\partial H}{\partial \lambda}$and obtains

$$\dot{\lambda_1} = \frac{\partial H}{\partial S} = \lambda_1[(-(1 + u_1)\beta_1 E - \beta_2 I - \beta_3 R - 1)] + \lambda_2[(1 + u_1)\beta_1 E + \beta_2 I + \beta_3 R]$$

$$\dot{\lambda_2} = \frac{\partial H}{\partial E} = (-\lambda_1(1 + u_1)\beta_1 S + \lambda_2((1 + u_1)\beta_1 S + (2 + u_1)\gamma$$

$$\dot{\lambda_3} = \frac{\partial H}{\partial I} = -\lambda_1 \beta_2 S + \lambda_2 \beta_2 S - \lambda_3(1 + f) + \lambda_4 f$$

$$\dot{\lambda_4} = \frac{\partial H}{\partial R} = -\lambda_1 \beta_3 S + \lambda_2 \beta_3 S - \lambda_4$$

Then, from equations that substitutes to equation$u^*(t)$which has been obtained the optimal form of for the solution control $u_1(t)$ is the vaccination effort given to reduce the number of susceptible individuals into latent individuals, $u_2(t)$ represents the control of the treatment given to the infected individual to be cured

$u_1{}^* = \min\left(\max\left(0, \frac{\beta_1 SE - \lambda_2 \beta_1 SE}{2C_1}\right), 1\right)$ and $u_2{}^* = \min\left(\max\left(0, \frac{\gamma E - \lambda_3 \gamma E}{2C_2}\right), 1\right)$.

4. Conclusions

From the discussion obtained some conclusions Model of malaria distribution in South Kalimantan as follows:

$$\frac{dS}{dt} = L - \beta_{11} SE - \beta_{21} SI - \beta_{31} SR - \mu S$$

$$\frac{dE}{dt} = \beta_{11}SE + \beta_{21}SI + \beta_{31}SR - \gamma_1 E - \mu E$$

$$\frac{dI}{dt} = \gamma_1 E - f_1 I - \mu I$$

$$\frac{dR}{dt} = f_1 I - \mu R$$

The lowest level of malaria infection occurs in Banjarmasin City because the value of reproduction numbers is essentially the smallest number. While the highest malaria infection rate occurred in Balangan Regency because the value of reproduction is essentially the largest number.

In the end of this research then the solutions will be found in form of control $u_1^* = \min\left(\max\left(0, \frac{\beta_1 SE - \lambda_2 \beta_1 SE}{2C_1}\right), 1\right)$ and $u_2^* = \min\left(\max\left(0, \frac{\gamma E - \lambda_3 \gamma E}{2C_2}\right), 1\right)$.

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